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Isotropization in Bianchi Type VII_h Vacuum Cosmology

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We find the most general solution of Einstein's equations for the Bianchi type VII_h vacuum case in the Brans–Dicke theory. For $w > 500$ the universe will become isotropic for any amount of initial anisotropy; for this model there is no inflationary expansion.

1. INTRODUCTION

The purpose of this paper is to give the most general solution for the Bianchi type VII_h vacuum case with total anisotropy ($R_1 \neq R_2 \neq R_3$) in the Brans–Dicke theory (BDT). It is shown that for a large value of the BDT coupling parameter w [present limits based on time-delay experiments require $w \geqq 500$ (Reasenberg *et al.*, 1979)] the universe will become isotropic for any amount of the initial anisotropy for the $h = 1$ case. For this model we do not have inflationary behavior.

2. BIANCHI TYPE VII_h FIELD EQUATIONS

The field equations for the Bianchi type VII_h vacuum model in which the metric is diagonalized (Lorentz-Petzold, 1984) are given by

$$\dot{H}_1 + 3HH_1 + \frac{1}{2R_3^2}\left[\left(\frac{R_1}{R_2}\right)^2 - \left(\frac{R_2}{R_1}\right)^2\right] + H_1(\ln \Phi)^{\cdot} - \frac{1}{2}\left[\left(\frac{R_1}{R_2}\right)^2 - \left(\frac{R_2}{R_1}\right)^2\right]\dot{\Omega}^2 - 2\left(\frac{h}{R_3}\right)^2 = 0 \qquad (1)$$

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$$\dot{H}_2 + 3HH_2 - \frac{1}{2R_3^2}\left[\left(\frac{R_1}{R_2}\right)^2 - \left(\frac{R_2}{R_1}\right)^2\right] + H_2(\ln\Phi)^{\cdot} + \frac{1}{2}\left[\left(\frac{R_1}{R_2}\right)^2 - \left(\frac{R_2}{R_1}\right)^2\right]\dot{\Omega}^2 - 2\left(\frac{h}{R_3}\right)^2 = 0 \tag{2}$$

$$\dot{H}_3 + 3HH_3 - \frac{1}{2R_3^2}\left(\frac{R_1}{R_2} - \frac{R_2}{R_1}\right)^2 + H_3(\ln\Phi)^{\cdot} - 2\left(\frac{h}{R_3}\right)^2 = 0 \tag{3}$$

$$H_1H_2 + H_1H_3 + H_2H_3 + 3H(\ln\Phi)^{\cdot} = \frac{1}{4R^6}\{12(hR_1R_2)^2 + (R_1^2 - R_2^2)^2[1 + (R_3\dot{\Omega})^2\} + \frac{w}{2}(\ln\Phi)^{\cdot 2} \tag{4}$$

$$\dot{\Omega} = 2h(2H_3 - H_1 - H_2)\left(\frac{R_1}{R_2} - \frac{R_2}{R_1}\right)^{-2} \tag{5}$$

$$(R^3\dot{\Phi})^{\cdot} = 0; \qquad (\cdot)^{\cdot} = \frac{d}{dt} \tag{6}$$

Here R_i $(i = 1, 2, 3)$ are the scale factors, $R^3 = R_1R_2R_3$, $H_i = \dot{R}_i/R_i$ are the Hubble parameters, $3H = \sum_{i=1}^{3} H_i$, $\Phi = \Phi(t)$ is the BDT scalar field, w is the coupling parameter of the BDT, $\Omega = \Omega(t)$ is the angle of rotation, and h is the family parameter of Bianchi type VII_h. By rescaling $d\Phi = R^{-3}\,dt$, one obtains equations (1)–(6) in the form

$$H_1' + \frac{H_1}{\Phi} + \frac{R_1^2R_2^2}{2}\left[\left(\frac{R_1}{R_2}\right)^2 - \left(\frac{R_2}{R_1}\right)^2\right] - \frac{1}{2}\left[\left(\frac{R_1}{R_2}\right)^2 - \left(\frac{R_2^2}{R_1}\right)\right]\Omega'^2 = 2h^2R_1^2R_2^2 \tag{7}$$

$$H_2' + \frac{H_2}{\Phi} - \frac{R_1^2R_2^2}{2}\left[\left(\frac{R_1}{R_2}\right)^2 - \left(\frac{R_2}{R_1}\right)^2\right] + \frac{1}{2}\left[\left(\frac{R_1}{R_2}\right)^2 - \left(\frac{R_2}{R_1}\right)^2\right]\Omega'^2 = 2h^2R_1^2R_2^2 \tag{8}$$

$$H_3' + \frac{H_3}{\Phi} - \frac{R_1^2R_2^2}{2}\left(\frac{R_1}{R_2} - \frac{R_2}{R_1}\right)^2 = 2h^2R_1^2R_2^2 \tag{9}$$

$$H_1H_2 + H_1H_3 + H_2H_3 - \frac{w}{2\Phi^2} + \frac{1}{\Phi}(3H) = \frac{1}{4}\left\{12(hR_1R_2)^2 + (R_1^2 - R_2^2)\left[1 + \left(\frac{\Omega'}{R_1R_2}\right)^2\right]\right\} \tag{10}$$

$$\Omega' = 2h(2H_3 - H_1 - H_2)\left(\frac{R_1}{R_2} - \frac{R_2}{R_1}\right)^{-2} \tag{11}$$

where $(\cdot)' = d/d\Phi$ and now $H_i = R_i'/R_i$. From equations (7) and (8) we have

$$\Phi[\Phi(\ln \Phi^2 R_1^2 R_2^2)']' = 4h^2 R_1^2 R_2^2 \Phi^2 \tag{12}$$

By introducing the new variables $r^2 = R_1^2 R_2^2 \Phi^2$ and $d\eta = d\Phi/\Phi$, we find for equation (12)

$$(\ln r)^{**} = 4h^2 r^2 \tag{13}$$

where $(\cdot)^* = d/d\eta$. Equation (13) can be integrated and we obtain

$$r = \frac{4CB^2}{\Phi^{-\sqrt{c}} - 16h^2 C\Phi^{\sqrt{c}}}; \qquad C, B > 0 \tag{14}$$

where c and B are constants of integration. From equations (7) and (8),

$$(H_1 - H_2)' + \frac{1}{\Phi}(H_1 - H_2) = -\left(1 - \frac{\Omega'^2}{R_1^2 R_2^2}\right) R_1^2 R_2^2 \left[\left(\frac{R_1}{R_2}\right)^2 - \left(\frac{R_2}{R_1}\right)^2\right] \tag{15}$$

Using the Lukash assumption (Lukash, 1974; Jantzen, 1980), $\Omega'^2 = R_1^2 R_2^2$, we can easily integrate equation (15),

$$\frac{R_1}{R_2} = \Phi^k \tag{16}$$

where k is a constant. From the definition of $r^2 = R_1^2 R_2^2 \Phi^2$ and (16) we obtain

$$R_1 = \frac{\sqrt{B}}{2h\Phi^{-(k+1)/2}(\Phi^{2k} - \Phi^{-2k})^{1/2}} \tag{17}$$

$$R_2 = \frac{\sqrt{B}}{2h\Phi^{(k+1)/2}(\Phi^{2k} - \Phi^{-2k})} \tag{18}$$

By substitution of equations (17) and (18) into (9) and using (11), finally we obtain the other scale factor R_3,

$$R_3 = \frac{\sqrt{B}\,(\Phi^k + \Phi^{-k})^{B/16h^3k}}{2h\Phi^{1/2}(\Phi^{2k} - \Phi^{-2k})^{1/2}} \tag{19}$$

with the corresponding Hubble parameters

$$H_1 = -\frac{1-k}{\Phi} - \frac{k}{\Phi}\frac{\Phi^{2k} + \Phi^{-2k}}{\Phi^{2k} - \Phi^{-2k}} \tag{20}$$

$$H_2 = -\frac{k+1}{\Phi} - \frac{k}{\Phi}\frac{\Phi^{2k} + \Phi^{-2k}}{\Phi^{2k} - \Phi^{-2k}} \tag{21}$$

$$H_3 = -\frac{1}{2\Phi} - \frac{k}{\Phi}\frac{\Phi^{2k} + \Phi^{-2k}}{\Phi^{2k} - \Phi^{-2k}} + \frac{B}{16h^3\Phi}\frac{\Phi^k - \Phi^{-k}}{\Phi^k + \Phi^{-k}} \tag{22}$$

By substitution of (17)–(19) into (10), we obtain the constants k and B,

$$k = \frac{B}{8h}; \qquad k = \left(\frac{2w+3}{11-4/h^2}\right)^{1/2} \tag{23}$$

3. DISCUSSION AND CONCLUSIONS

The BDT is consistent with the local observations in the solar system as long as the coupling parameter w is about equal to or greater than 500 (Reasenberg *et al.*, 1979); from equation (23) for $w > 500$ for real constants we have that $B = 96$, $h = 1$, and $k = 12$. Making a plot for R_1, R_2, and R_3 [equations (17)–(19)] versus the "temporal" parameter Φ (Fig. 1), we can see that we have physical solutions for $\Phi > 1$ ($\Phi < 1$ implies that R_i becomes imaginary), that is, $R_i \to +\infty$; there is no singularity, the universe is infinite in size, and for $\Phi \to \infty$ the scale factors become equal to 0, that is, we have the final singularity.

For the Hubble parameters H_1, H_2, and H_3 [equations (20)–(22) and Fig. 2), we see that for any amount of initial anisotropy, that is, $k \neq 0$, the universe tends to be isotropized very fast ($H_1 = H_2 = H_3$ when $\Phi \to \infty$).

This model does not have inflationary behavior, because for short values of Φ ($\Phi < 1$) the scale factors are imaginary. For $\Phi \to 1$, $R_1 = R_2 = R_3$ (Fig. 1), that is, isotropy.

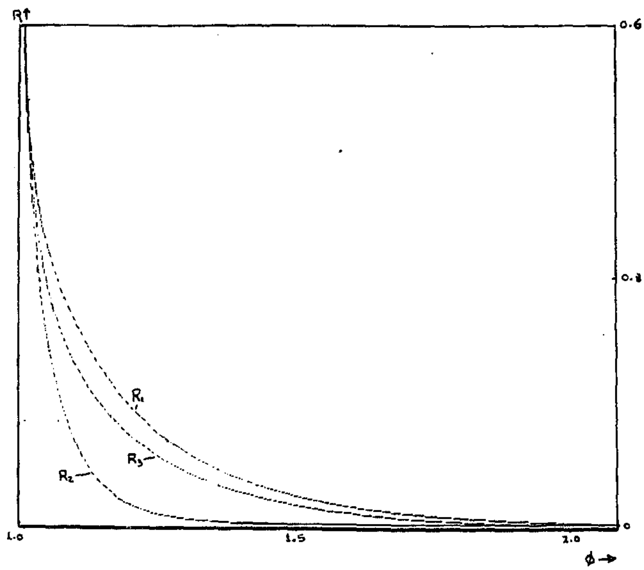

Fig. 1. "Time" dependence of the scale factors R_1, R_2, and R_3 versus the scalar field Φ. R_1, R_2, and R_3 are in units of $(B/2)^{1/2}$ ($B = 96$, $h = 1$, $k = 12$, $w > 500$).

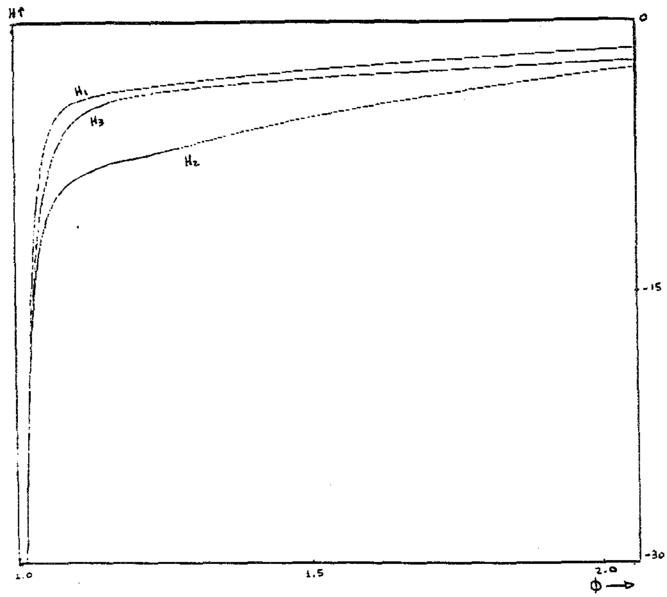

Fig. 2. "Time" dependence of the Hubble parameters H_1, H_2, and H_3 versus Φ. For $\Phi > 2$ the universe tends to be isotropized, $H_1 \cong H_2 \cong H_3$ (H_3 is not in the same scale).

In conclusion, we have found the general vacuum solution for the Bianchi type VII$_h$ case in BDT, and showed that for $w > 500$ the universe tends to be isotropized for the case of a Bianchi type VII$_h$ perfect fluid solution analyzed by us (Guzmán, 1989). In future papers we will discuss other Bianchi models in BDT.

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